Lecture 18: Regularity & Fourier Coefficients

- In our theorem on Uniform Convergence, we computed
$$C_k[f'] = ik\, C_k[f] \quad \text{for } f \in C^1(\mathbb{T}).$$

Repeating this gives

$\boxed{\text{Lemma 8.9}}$ For $f \in C^m(\mathbb{T})$, $C_k[f^{(m)}] = (ik)^m C_k[f]$

- We will notice that regularity is tied to the decay of the coefficients. Indeed, the above says $\sum |(ik)^m C_k[f]|^2 < \infty$.

Thus, we introduce some notation. For $\alpha \in \mathbb{R}$,

$a_k = o(k^\alpha)$ means $\lim\limits_{k \to \infty} \dfrac{|a_k|}{|k^\alpha|} = 0$

$a_k = O(k^\alpha)$ means $|a_k| \leq C|k|^\alpha$ for large k (or $\lim\limits_{k\to\infty} |a_k|/|k^\alpha| < \infty$)

and some $C > 0$ independent of k.

$\boxed{\text{Th}^{\underline{m}}\ 8.10}$ For $f \in C^m(\mathbb{T})$ with $m \in \mathbb{N}_0$, $\sum\limits_{k \in \mathbb{Z}} k^{2m} |C_k[f]|^2 < \infty$.

$\boxed{\text{Pf}}$ Since $f \in C^m(\mathbb{T})$, Bessel's Inequality Applies on $f^{(m)}$

so $\sum |(ik)^m C_k[f]|^2 = \sum k^{2m} |C_k[f]|^2 < \infty$
$$\overset{\shortparallel}{\sum |C_k[f']|^2} \quad \square$$

Rmk: This shows $C_k[f] = o(k^{-m})$

Ex.) For $h(x) = 3\pi x^2 - 2x^3$ on $(0, \pi)$, we computed

$$C_k[h] = \begin{cases} \pi^3/2 & k=0 \\ -24/\pi k^4 & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

~~So~~ $C_k[h] \in O(k^{-4})$.

Since $h(x)$ is C^2 on $\mathbb{T}$ (the extension to $\mathbb{T}$ as an even function), we only are guaranteed $C_k[h] \in o(k^{-2})$, So this is better than expected.

- If we wish to apply these tools to PDEs, we aim for a converse that tells us when a function is C^m. In essence, rapid decay of $C_k[f]$ gives $S_n[f] \to f$ in many ways. Since $S_n[f] \in C^\infty$, the type of convergence tells us about the regularity of f.

$\boxed{\text{Th}^{\underline{m}}}$ $\boxed{8.12}$ Suppose $f \in L^2(\mathbb{T})$ has $\sum_{k \in \mathbb{Z}} |k^m C_k[f]| < \infty$ (A) for $m \in \mathbb{N}_0$. Then, $f \in C^m(\mathbb{T})$.

$\boxed{\text{Pf}}$ If $m=0$, $\sum_{k \in \mathbb{Z}} |C_k[f]| < \infty$ gives $S_n[f] \to f$ uniformly, So $f \in C^0$ as we proved before.

Consider $m=1$ and Set $f_n(x) = S_n[f](x)$. Notice

$$f_n'(x) = \sum_{k=-n}^{n} ik c_k e^{ikx} \quad \text{for } c_k = C_k[f]$$

Further, by the same $m=0$ case, f_n' converges uniformly to $g \in C^0(\mathbb{T})$. We show $g = f'$! To see this, notice

$$\frac{f(x+t)-f(x)}{t} - g(x) = \left[\frac{f_n(x+t)-f_n(x)}{t} - f_n'(x)\right] + (f_n'(x) - g(x)) + R_n(x,t)$$

as $t \to 0$, the first term $\to 0$. As $n \to \infty$, the second $\to 0$.

Lastly, $R_n(x,t) = \sum_{|k|>n} c_k \dfrac{e^{ik(x+t)} - e^{ikx}}{t}$ which converges for $t \neq 0$ by (A), and converges absolutely.

Since $|R_n(x,t)| \leq \sum_{|k|>n} |c_k| \cdot \left|\frac{e^{ikt}-1}{t}\right| \leq \sum_{|k|>n} |c_k| \left|\frac{2\sin(kt/2)}{t}\right|$

$$\leq \sum_{|k|>n} |c_k| \cdot |k| \qquad \text{(Taylor Approx.)}$$

$|R_n(x,t)| \to 0$ as $n \to \infty$ by (A).

To formalize the convergence, pick $\varepsilon > 0$. Pick large n so $|R_n(x,t)| < \varepsilon/3$ and $|f_n'(x) - g(x)| < \varepsilon/3$

Pick $\delta > 0$ so for $|t| < \delta$,

$$\left| \frac{f_n(x+t) - f_n(x)}{t} - f_n'(t) \right| < \varepsilon/3 .$$

Then, for such $|y| < \delta$, $\left| \frac{f(x+y) - f(x)}{y} - g(x) \right| < \varepsilon$.

Hence, $f' = g$ and $f \in C^1(\mathbb{T})$. Repeating this gives higher m. $\square$

- We also now have the tools to prove what we originally desired.

$\boxed{\text{Th}^{\underline{m}}}$ For $h \in C^0(\mathbb{T})$, the heat equation $\begin{cases} \partial u/\partial t - \partial^2 u/\partial x^2 = 0 \\ \lim_{t \to 0} u(t,x) = h(x) \end{cases}$ admits a solution $u \in C^\infty((0,\infty) \times \mathbb{T})$ defined for $t > 0$ by

$$u(t,x) = \sum_{k \in \mathbb{Z}} c_k[h] e^{-k^2 t} e^{ikx}$$

$\boxed{\text{Pf}}$ For $t > 0$, $c_k[h]$ decay more slowly than $c_k[h]e^{-k^2 t}$ and in $C^\infty(\mathbb{T})$. The same applies to t-derivatives. So $u(t,\cdot)$ is defined and in Let $u_n(x) = \sum_{-n}^{n} c_k[h] e^{-k^2 t} e^{ikx}$ and $\frac{\partial u_n}{\partial t} = \sum_{-n}^{n} (-k^2) c_k[h] e^{-k^2 t} e^{ikx}$. Since

~~[illegible]~~ $c_k[h] = o(k^{-1})$,

$$|-k^2 c_k[h] e^{-k^2 t} e^{ikx}| \leq C_h k e^{-k^2 t}$$

as $n \to \infty$, $\partial u_n/\partial t$ then converges absolutely for $t > 0$

and we may define $g = \lim_{n\to\infty} \partial u_n/\partial t$. If we fix $\varepsilon > 0$ and focus on $t \geq \varepsilon$, the convergence is uniform, and so g is continuous for $t \geq \varepsilon$. Hence, let $\varepsilon \to 0$ and $g \in C^0((0,\infty) \times \mathbb{T})$.

Exactly as in the previous thm, we may argue $g = \partial u/\partial t$, and acting on higher derivatives, $u \in C^\infty((0,\infty) \times \mathbb{T})$. Furthermore, since each u_n satisfies the wave eqn, $u(t,x)$ does, since $u_n \to u$ & $\partial u_n/\partial t \to \partial u/\partial t$ uniformly in $(\varepsilon, \infty) \times \mathbb{T}$ for any $\varepsilon > 0$.

To show the limit, we need the Fourier Transform (later). $\square$

$\rightarrow$ Rescaling $[0,\ell] \to [0,\pi]$ & extending functions connects $[0,\ell]$ to $\mathbb{T}$